

# Three-dimensional Background Field Gravity: A Hamilton-Jacobi analysis

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## Abstract

We analyse the constraint structure of the Background Field model for three dimensional gravity including a cosmological term via the Hamilton-Jacobi formalism. We find the complete set of involutive Hamiltonians that assures the integrability of the system and calculate the characteristic equations of the system. We established the equivalence between these equations and the field equations and also obtain the generators of canonical and gauge transformations.

*Keywords:* Constrained Systems, Hamilton-Jacobi formalism, Background Field model.

## 1 Introduction

Topological Quantum Field Theories (TQFT) were introduced by Witten [1] at the late 80s and until now they have found a wide range of applications in Physics. One characteristic of these theories is that their correlation functions do not depend on the space-time metric. According to Birmingham [2] the TQFT can be divided into two groups: the Witten (or cohomological) type and the Schwarz type. The Chern-Simons (CS) gauge theory is a Schwarz type TQFT defined in odd dimensions which is used, for example, in addition with three-dimensional kinetic actions to build the so called Topologically Massive Theories [3].

The Background Field (BF) model is another Schwarz type TQFT and had been widely used due to its relation with Gravity. For example, there has been shown that the two-dimensional BF model can be equivalent to the two-dimensional Jackiw-Teitelboim Gravity [4] for a given gauge group [5]. The three-dimensional BF model is equivalent to the first order formulation of pure General Relativity under the Lorentz gauge group  $SO(2, 1)$  [6] and the four-dimensional gravity is equivalent to the Plebanski action [7] which consists in a BF action plus a Lagrangian multiplier. An extensive review between these equivalences can be found in [8].

The BF lower dimensional models of gravity are good laboratories for the study of spin foam quantization [9] and loop quantum gravity. In both schemes of quantization, the symplectic structure of the BF model is of utmost importance. In order to identify the correct phase space, the Dirac canonical analysis [10] is one of the most used tools. This analysis has been done in two [11] and three [12] dimensional BF models of gravity. Nonetheless,

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there are other schemes of constraint analysis, as the Faddeev-Jackiw [13] formalism and the Hamilton-Jacobi (HJ) formalism.

A first attempt to use the HJ formalism as an approach to constrained systems was given by Dominici, et. al. [14]. Here we will deal with the approach developed by Güler [15] as an extension to the Carathéodory's equivalent Lagrangians method to the calculus of variations [16]. The conditions for stationary action are reduced to a set of Hamilton-Jacobi partial differential equations, also called Hamiltonians, that must obey the Frobenius' Integrability Condition (IC). In [17] has been shown that in order to satisfy the IC the non-involutive Hamiltonians must be eliminated, this way they redefine the dynamic of the system by building the Generalised Bracket (GB). Therefore, we end with a set of complete involutive Hamiltonians, which plays the role of generators of the canonical transformations [18]. The Hamilton-Jacobi formalism has been generalised to higher order Lagrangians and Berezin systems, among others [19], as well as applied to different kind of physical systems, more recently to Topologically Massive theories [20] and gravity models [21], including the two-dimensional BF gravity [22]. In this article we will apply the HJ formalism to the three-dimensional BF model for gravity.

In the following section we will shown the HJ formalism (for a more detailed explanation see [17][18]). In section 3 the three-dimensional BF gravity will be presented. In section 4 we will perform its Hamilton-Jacobi constraint analysis and build the Generalised Brackets. In section 5 we will compute the characteristic equations (CE) and analyse the dynamical evolution along the independent parameters of the theory. From this analysis we obtain the equivalence between the Lagrangian equations of motion and the temporal evolution of the CE. From the evolution along the parameters related the involutive Hamiltonians, we obtain the generators of canonical and gauge transformations. In section 6 we will discuss the results.

## 2 The Hamilton-Jacobi Formalism

Let us consider a physical system with a Lagrangian function  $L = L(x^i, \dot{x}^i, t)$ , where the Latin indices  $i, j$  go from 1 to  $n$ , being  $n$  the dimension of the configuration space. This Lagrangian is called singular or constrained if it does not satisfy the Hessian Condition, which states that the matrix elements  $W_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$  has a determinant equal to zero. Whenever the Hessian Condition ( $\det W_{ij} \neq 0$ ) is not satisfied, it is implied that some of the conjugated momenta  $p_i = \frac{\partial L}{\partial \dot{x}^i}$  are not invertible on the velocities. By considering  $k$  non-invertible momenta and  $m = n - k$  invertible momenta, we have

$$p_z - \frac{\partial L}{\partial \dot{x}^z} = 0, \quad (1)$$

where  $z = 1, \dots, k$ . Defining  $H_z \equiv -\frac{\partial L}{\partial \dot{x}^z}$ , the above equation is rewritten as

$$H'_z \equiv p_z + H_z = 0. \quad (2)$$

We call *Hamiltonians* the constraints represented in this way. Defining  $p_0 \equiv \frac{\partial S}{\partial t}$ , the HJ equation is the Hamiltonian

$$H'_0 \equiv p_0 + H_0 = 0. \quad (3)$$

The canonical Hamiltonian function  $H_0 = p_a \dot{x}^a + p_z \dot{x}^z - L$ , regarding  $a = 1, \dots, m$ , does not depend on the non-invertible velocities  $\dot{x}^z$  if the constraints are carried out. Putting

together (2) and (3), we form the initial set of Hamilton-Jacobi Partial Differential Equations (HJPDE):

$$H'_\alpha \equiv p_\alpha + H_\alpha = 0, \quad (4)$$

where  $\alpha = 0, 1, \dots, k$ . Through the Cauchy's method [16], the characteristic equations related to the first order equations system (4) are given by

$$dx^a = \frac{\partial H'_\alpha}{\partial p_a} dt^\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial x^a} dt^\alpha, \quad dS = (p_a dx^a - H_\alpha dt^\alpha). \quad (5)$$

From these differential equations, the Poisson Brackets (PB) defined on the extended phase space  $(x^a, t^\alpha, p_a, p_\alpha)$  can be used to express in a concise form the evolution of any function  $f = f(x^a, t^\alpha, p_a, p_\alpha)$ :

$$df = \{f, H'_\alpha\} dt^\alpha. \quad (6)$$

This is the *fundamental differential* whereby the Hamiltonians can be seen as the generators of the dynamical evolution of the phase space functions.

A geometrical interpretation can be given at this point. The solutions of the first two equations of (5) give rise to a congruence of curves on the reduced phase space  $(x^a, p_a)$ . The characteristic curves  $x^a(t, x^z)$  describe the dynamical trajectories and depend on the  $k + 1$  parameters  $t^\alpha$  which in turn must be regarded as the *independent variables* of the system. A complete solution of (4) is given by a family of surfaces orthogonal to the characteristic curves and its existence is ensured by satisfying the Frobenius' integrability condition [18] which is written as

$$\{H'_\alpha, H'_\beta\} = C_{\alpha\beta}{}^\gamma H'_\gamma. \quad (7)$$

It means the Hamiltonians must close a *Lie algebra*. Equivalently,

$$dH'_\alpha = 0. \quad (8)$$

Hamiltonians that satisfy the Frobenius integrability condition are called *involutive* while the *non-involutive* are those that do not satisfy it. We can add new constraints to the system imposing condition (8) and then completing the set of HJPDE. However, sometimes this procedure is not sufficient to make the set of HJPDE integrable. When the condition (8) is imposed, some Hamiltonians may provide relations that exhibit dependence between some parameters. These Hamiltonians can be used to construct a new algebra which we call the *Generalised Brackets* (GB):

$$\{A, B\}^* = \{A, B\} - \{A, H'_a\} (M_{\bar{a}\bar{b}})^{-1} \{H'_b, B\}.$$

The indices  $\bar{a}$  and  $\bar{b}$  are related to the non-involutive Hamiltonians whose parameters are somehow related. The matrix  $M$  is built from the PB of these Hamiltonians, i.e., its elements are  $M_{\bar{a}\bar{b}} = \{H'_a, H'_b\}$ . In this way, these non-involutive Hamiltonians are absorbed in the new algebra. The integrability of the remaining Hamiltonians must be analysed through the GB algebra instead of the PB algebra. New Hamiltonians may be added to complete the HJPDE set in this process until we get as result an integrable set of HJPDE.

Let us define the variables on the extended phase space as  $z^I = (x^a, t^\alpha, p_a, p_\alpha)$  and define the vector field  $X_\alpha$  with components

$$X_\alpha^I \equiv \{z^I, H'_\alpha\}^*, \quad (9)$$

such that, any function on the extended phase space can be written as

$$dF = \{F, H'_\alpha\}^* dt^\alpha = X_\alpha [F] dt^\alpha. \quad (10)$$

The vector  $X_\alpha$  are related to the dynamical evolution of the system, since the CE are included on (10). From the definition of vectors  $X_\alpha$  and using the Jacobi Identity we obtain

$$[X_\alpha, X_\beta] F = \{\{F, H'_\beta\}^*, H'_\alpha\}^* - \{\{F, H'_\alpha\}^*, H'_\beta\}^* = \{\{H'_\alpha, H'_\beta\}^*, F\}^*, \quad (11)$$

Whenever the system is integrable, i.e., (7) or (8) are valid, we can write

$$[X_\alpha, X_\beta] F = -\{f^\gamma_{\alpha\beta} H'_\gamma, F\}^* = f^\gamma_{\alpha\beta} X_\gamma [F] - \{f^\gamma_{\alpha\beta}, F\}^* H'_\gamma, \quad (12)$$

where  $f^\gamma_{\alpha\beta} = -C^\gamma_{\alpha\beta}$ . If the structure constants are independent of the variables of the extended phase space the IC becomes a condition over the commutator

$$[X_\alpha, X_\beta] = f^\gamma_{\alpha\beta} X_\gamma, \quad (13)$$

which is, indeed, the necessary condition for  $X_\alpha$  to be a complete basis.

In general, a transformation of a function  $F$  can be written as

$$\delta F = \delta t^\alpha X_\alpha F, \quad (14)$$

where  $\delta t^\alpha = \bar{t}^\alpha - t^\alpha$  are arbitrary functions of  $z^I$ . However, notice that if we choose  $\delta t^\alpha = dt^\alpha$ , equation (14) becomes the fundamental differential. For any variable of the extended phase space  $z^I$  we have

$$\delta z^I = \bar{z}^I(\bar{t}^\alpha) - z^I(t^\alpha) = \delta t^\alpha X_\alpha [z^I]. \quad (15)$$

Now, let us consider a transformation  $g$  such that

$$\bar{z}^I(\bar{t}^\alpha) = g z^I(t^\alpha). \quad (16)$$

In this case

$$g = 1 + \delta t^\alpha X_\alpha. \quad (17)$$

We say that transformation  $g$  carries the infinitesimal flows generated by the vectors  $X_\alpha$ . This is what we call characteristics flows (CF). It can be shown that whenever the IC is satisfied, the transformation  $g$  has an inverse

$$g^{-1} = 1 - \delta t^\alpha X_\alpha. \quad (18)$$

and also preserve the symplectic structure  $\omega \equiv dx^a \wedge dp_a + dt^\alpha \wedge dp_\alpha + dH_\alpha \wedge dt^\alpha$

$$g\omega g^{-1} = \omega. \quad (19)$$

This show that  $g$  are canonical transformations and that the complete set of involutive Hamiltonians  $H'_\alpha$  are the generators of these transformations.

In order to relate the canonical transformations with the gauge ones, we need to restrict the study to fixed times  $\delta t^0 = \delta t = 0$ , which is the classical equivalent to a fixed point transformation in field theory. The transformation on any variable  $z^I$  now reads

$$\delta z^I = \{z^I, H'_z\}^* \delta t^z, \quad (20)$$

If we can keep this transformation canonical, the IC must be satisfied, this is

$$\{H'_x, H'_y\}^* = C^z_{xy} H'_z, \quad (21)$$

Nonetheless, this condition does not guarantee the integrability on the algebra of the Hamiltonians, which is

$$\{H'_x, H'_y\}^* = C^0_{xy} H'_0 + C^z_{xy} H'_z. \quad (22)$$

To conciliate both equations we must consider whether  $C^0_{xy} = 0$  or  $H'_0 = 0$ . However, condition  $C^0_{xy} = 0$  is too strong since it implies that  $\{H'_0, H'_z\} = 0$ , which is almost never satisfied. On the other hand, the condition  $H'_0 = 0$  constrains the phase space. Under this assumption, we define

$$G^{can} \equiv H'_z \delta t^z, \quad (23)$$

which is the generator of the canonical transformations, once that

$$\delta z^I = \{z^I, G^{can}\}^*. \quad (24)$$

### 3 Three-Dimensional BF model

Let us consider a  $d$ -dimensional manifold  $\mathcal{M}$ , a Lie group  $G$ , a connection  $A$  and a  $(d-2)$ -form  $B$  called Background Field. With those elements let us build the following action

$$W_{BF} = \int_{\mathcal{M}} tr[B \wedge F], \quad (25)$$

where  $F$  is the curvature of the connection  $A$ , i.e.,  $F = DA$ . Due to the properties of the trace and the exterior product  $\wedge$  it is straightforward to see that this action is gauge invariant.

In three dimensions we can add another invariant  $tr[B \wedge B \wedge B]$ . Therefore, the three-dimensional BF action can be written as

$$W_{BF} = \int_{\mathcal{M}} tr(B \wedge F(A) + \kappa B \wedge B \wedge B). \quad (26)$$

where  $\kappa$  is a constant. Due to its construction, (26) is invariant under gauge transformation:

$$\delta A = D\chi, \quad \delta B = [B, \chi], \quad (27)$$

but also quasi-invariant under shift transformation:

$$\delta B = D\eta, \quad \delta A = 3\kappa[B, \eta], \quad (28)$$

being  $\xi$  and  $\eta$  arbitrary functions.

It has been shown that in three dimensions and, considering  $G$  as the Lorentz group  $SO(1,2)$ , the BF action (26) is equivalent to Einstein-Hilbert-Palatini gravity in terms of vielbeins. Therefore, considering  $G$  as the Lorentz group and  $\kappa = -\Lambda/3$ , the action (26) represents Riemann gravity plus Cosmological constant.

Before proceed with any kind of quantization scheme, the reduced phase space of the system must be well defined. The determination of the true degrees of freedom are determined after the analysis of the constrains of the theory.

## 4 The Hamilton-Jacobi analysis of the 3D BF gravity

The constraint analysis is not covariant. We refer to one specific time choice to build the HJ equations. It is, then, appropriated to leave the differential forms notation and write the Lagrangian in terms of the components of the background and gauge field, i.e.,

$$A = A_\mu^a J_a dx^\mu, \quad B = B_\mu^a J_a dx^\mu, \quad (29)$$

where  $J_a$  are generators of the  $G = SO(1, 2)$  group. These generators satisfy  $[J_a, J_b] = f_{abc} J_c$  and  $\text{tr}(J_a J_b) = \frac{1}{2} \eta_{ab}$ , where  $\eta_{ab} = \text{diag}(+, -, -)$ . Therefore

$$\mathcal{L} = \frac{1}{2} \epsilon^{\mu\gamma\nu} (B_{a\mu} F_{\gamma\nu}^a - \frac{\Lambda}{3} f_{abc} B_\mu^a B_\gamma^b B_\nu^c), \quad (30)$$

where  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c$ . The equations of motion are

$$0 = \epsilon^{\mu\gamma\nu} (F_{\gamma\nu}^a - \Lambda f_{bc}^a B_\gamma^b B_\nu^c), \quad (31)$$

$$0 = \epsilon^{\mu\gamma\nu} D_\gamma B_\nu^a. \quad (32)$$

Here we had made use of the definition of covariant derivative

$$D_\mu \theta_\nu^a \equiv \partial_\mu \theta_\nu^a + f_{bc}^a A_\mu^b \theta_\nu^c. \quad (33)$$

Furthermore, equation (31) represent the dynamical equation of three-dimensional gravity, and (32) represent the zero torsion condition.

Now, to begin with the HJ analysis of the three-dimensional BF gravity, we compute the momenta  $\pi^a$  and  $\Pi^a$  conjugated to  $A_\mu^a$  and  $B_\mu^a$  respectively

$$\pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu^a} = \epsilon^{0\mu\nu} B_{a\nu}, \quad (34)$$

$$\Pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_0 B_\mu^a} = 0. \quad (35)$$

The expressions above do not depend on any velocities  $\partial_0 A_\mu^a$ ,  $\partial_0 B_\mu^a$ . Therefore they are canonical constraints of the theory. It turns out the canonical Hamiltonian density is given by

$$\mathcal{H}_0 = -\epsilon^{0\gamma\nu} [A_{a0} D_\gamma B_\nu^a + B_{a0} (F_{\gamma\nu}^a - \Lambda f_{bc}^a B_\gamma^b B_\nu^c)]. \quad (36)$$

Let us define  $\pi \equiv \partial_0 S$ . Then, the initial set of HJPDE is

$$\mathcal{H}' \equiv \pi + \mathcal{H}_0 = 0, \quad (37)$$

$$\mathcal{A}_a'^0 \equiv \pi_a^0 = 0, \quad (38)$$

$$\mathcal{A}_a'^1 \equiv \pi_a^1 - B_{a2} = 0, \quad (39)$$

$$\mathcal{A}_a'^2 \equiv \pi_a^2 + B_{a1} = 0, \quad (40)$$

$$\mathcal{B}_a'^\mu \equiv \Pi_a^\mu = 0. \quad (41)$$

The first Hamiltonian  $\mathcal{H}'$  is associated with the time parameter  $t \equiv x_0$ . The Hamiltonians  $\mathcal{A}_a'^\mu$  arose from the non-invertible momenta  $\pi_a^\mu$  and are related to the parameters  $\lambda_\mu^a \equiv A_\mu^a$ . Analogously, the Hamiltonians  $\mathcal{B}_a'^\mu$  are referred to the parameters  $\epsilon_\mu^a \equiv B_\mu^a$ .

The fundamental PB of the model are

$$\{A_\mu^a(x), \pi_b^\nu(x')\} = \delta_b^a \delta_\mu^\nu \delta^2(\mathbf{x} - \mathbf{x}'), \quad (42)$$

$$\{B_\mu^a(x), \Pi_b^\nu(x')\} = \delta_b^a \delta_\mu^\nu \delta^2(\mathbf{x} - \mathbf{x}'). \quad (43)$$

The fundamental differential characterizes the evolution of any function of the phase space. It is expressed as

$$df(x) = \int (\{f(x), \mathcal{H}'(x')\} dt + \{f(x), \mathcal{A}_a^\mu(x')\} d\lambda_\mu^a + \{f(x), \mathcal{B}_a^\mu(x')\} d\epsilon_\mu^a) d^2x'. \quad (44)$$

Now we check the integrability of the HJPDE. When the IC is applied to the Hamiltonians  $\mathcal{A}_a^1$ ,  $\mathcal{A}_a^2$ ,  $\mathcal{B}_a^1$  and  $\mathcal{B}_a^2$  we get relations of dependence between the parameters related to them. This information tells us that these Hamiltonians are non-involutive and can be used to construct the GB.

Let us rename  $h_a^0 \equiv \mathcal{A}_a^1$ ,  $h_a^1 \equiv \mathcal{A}_a^2$ ,  $h_a^2 \equiv \mathcal{B}_a^1$  and  $h_a^3 \equiv \mathcal{B}_a^2$ . Let us denote  $I, J = 0, 1, 2, 3$  as the indices of the elements of the matrix  $M_{ab}^{IJ}(x, y) \equiv \{h_a^I(x), h_b^J(y)\}$ . We have

$$M(x, y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \delta^{ab} \delta^2(\mathbf{x} - \mathbf{x}').$$

This matrix has inverse

$$M^{-1}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \delta^{ab} \delta^2(\mathbf{x} - \mathbf{x}'),$$

with this inverse we define the GB as

$$\{f(x), g(x')\}^* = \{f(x), g(x')\} - \int \{f(x), h_c^I(y)\} [M^{-1}(y, y')]_{IJ}^{cd} \{h_d^J(y'), g(x')\} dy dy'.$$

We can use this expression to find the fundamental GB of the theory. The non-vanishing results are given bellow:

$$\{A_\mu^a(x), \pi_b^\nu(x')\}^* = \delta_b^a \delta_\mu^\nu \delta^2(\mathbf{x} - \mathbf{x}'), \quad (45)$$

$$\{B_0^a(x), \Pi_b^0(x')\}^* = \delta_b^a \delta^2(\mathbf{x} - \mathbf{x}'), \quad (46)$$

$$\{A_\mu^a(x), B_\nu^b(x')\}^* = \delta^{ab} \epsilon_{0\mu\nu} \delta^2(\mathbf{x} - \mathbf{x}'). \quad (47)$$

Comparing with the original PB (42,43), we notice that  $B_a^i, \Pi_i^a$  are no longer conjugated variables. In fact, the  $B_a^1$  now plays the role of  $-\pi_a^2$  and  $B_a^2$  the role of  $\pi_a^1$ . Only  $B_0^a(x), \Pi_a^0$  and  $A_\mu^a, \pi_b^\nu$  remains as conjugated variables.

After building the GB, the fundamental differential (44) now takes the form

$$df(x) = \int (\{f(x), \mathcal{H}'(x')\}^* dt + \{f(x), \mathcal{A}_a^{\prime 0}(x')\}^* d\lambda_0^a + \{f(x), \mathcal{B}_a^{\prime 0}(x')\}^* d\epsilon_0^a) d^2x'. \quad (48)$$

We still need to analyse the IC of the Hamiltonians  $\mathcal{A}_a^{\prime 0}$  and  $\mathcal{B}_a^{\prime 0}$ . By imposing  $d\mathcal{A}_a^{\prime 0} = 0$  and  $d\mathcal{B}_a^{\prime 0} = 0$  we notice that we need to introduce two new Hamiltonians:

$$\mathcal{C}^{\prime a} \equiv \epsilon^{0\gamma\nu} D_\gamma B_\nu^a = 0, \quad (49)$$

$$\mathcal{D}^{\prime a} \equiv \frac{1}{2} \epsilon^{0\gamma\nu} [F_{\gamma\nu}^a - \Lambda f_{bc}^a B_\gamma^b B_\nu^c] = 0. \quad (50)$$

Notice that the canonical Hamiltonian (36) now can be written as  $\mathcal{H}_0 = -A_{a0}\mathcal{C}'^a - B_{a0}\mathcal{D}'^a$ . The fields  $\mathcal{A}_a'^0$  and  $\mathcal{B}_a'^0$  have the role of Lagrange multipliers since they are coefficients of the constraints in the canonical Hamiltonian. The new constraints also satisfy the IC and there is no need to introduce new constraints or redefine the algebra. The integrability programme is then achieved and the complete set of involutive Hamiltonians is  $\mathcal{A}_a'^0, \mathcal{B}_a'^0, \mathcal{C}'^a, \mathcal{D}'^a$ .

Let us define

$$\mathcal{C}'^a(\alpha) \equiv \int \alpha(y) \mathcal{C}'^a(y) d^2y, \quad (51)$$

$$\mathcal{D}'^a(\beta) \equiv \int \beta(y) \mathcal{D}'^a(y) d^2y, \quad (52)$$

where  $\alpha$  and  $\beta$  are weight functions. It follows the relation

$$\{\mathcal{C}'^a(\alpha_1), \mathcal{C}'^b(\alpha_2)\}^* = f^{ab}_c \mathcal{C}'^c(\alpha_1, \alpha_2), \quad (53)$$

$$\{\mathcal{C}'^a(\alpha_1), \mathcal{D}'^b(\beta_1)\}^* = f^{ab}_c \mathcal{D}'^c(\alpha_1, \beta_1), \quad (54)$$

$$\{\mathcal{D}'^a(\beta_1), \mathcal{D}'^b(\beta_2)\}^* = -\Lambda f^{ab}_c \mathcal{C}'^c(\beta_1, \beta_2). \quad (55)$$

Note that for  $\Lambda = 0$ , i.e., the pure three-dimensional gravity, the Hamiltonians satisfy the Poincaré algebra  $ISO(2, 1)$ , we also identify  $\mathcal{D}'^a$ , which now commute with all the other Hamiltonians as the generator of translations. For  $\Lambda \neq 0$ , the Hamiltonians close the  $AdS$  or  $dS$  algebra.

## 5 Characteristic Equations of the 3D BF Gravity

The IC allows us to find the complete set of involutive Hamiltonians:  $\mathcal{A}_a'^0, \mathcal{B}_a'^0, \mathcal{C}'^a, \mathcal{D}'^a$ , all of them play a role in the evolution of the systems and must be added in the fundamental differential. Let us rename

$$\begin{aligned} \mathcal{H}_a'^0 &\equiv \mathcal{A}_a'^0 &\longrightarrow \omega_a^0, \\ \mathcal{H}_a'^1 &\equiv \mathcal{B}_a'^0 &\longrightarrow \omega_a^1, \\ \mathcal{H}_a'^2 &\equiv \mathcal{C}'^a &\longrightarrow \omega_a^2, \\ \mathcal{H}_a'^3 &\equiv \mathcal{D}'^a &\longrightarrow \omega_a^3, \end{aligned}$$

where the  $\omega_a$  are the respective parameters. The final form of the fundamental differential is

$$df(x) = \int dx' \left( \{f(x), \mathcal{H}'(x')\}^* dt + \sum_{\kappa=0}^3 \{f(x), \mathcal{H}'^\kappa_a(x')\}^* d\omega_a^\kappa \right). \quad (56)$$

The CE are obtained from (56), by evaluating  $f$  for the fields  $(A_\mu^a, B_\mu^a)$  and the momenta  $(\pi_a^\mu, \Pi_a^\mu)$ . For the first set we have

$$dA_\mu^a = \delta_\mu^0 \delta^{ab} d\omega_b^0 + \delta_\mu^i \left[ (D_i A_0^a - \Lambda f_{bc}^a B_i^b B_0^c) dt - \delta^{ab} D_i d\omega_b^2 + \Lambda f_{bc}^a B_i^c d\omega_b^3 \right], \quad (57)$$

$$dB_\mu^a = \delta_\mu^0 d\omega^{a1} + \delta_\mu^i \left[ (D_i B_0^a - f_{bc}^a A_0^b B_i^c) dt - f^{ab}_c B_i^b d\omega_c^2 - \delta^{ab} D_i d\omega_b^3 \right], \quad (58)$$

and

$$\begin{aligned} d\pi_a^\mu &= \epsilon^{0\gamma\rho} \left[ \delta_0^\mu D_\gamma B_{a\rho} - \delta_\rho^\mu (D_\gamma B_{a0} - f_a^{bc} A_{b0} B_{c\gamma}) \right] dt \\ &\quad + \epsilon^{0\gamma\rho} \left\{ \delta_\gamma^\mu f_a^{bc} B_{c\rho} d\omega_b^2 + \delta_\gamma^\mu \delta_a^b D_\gamma d\omega_b^3 \right\}, \end{aligned} \quad (59)$$

$$d\Pi_a^\mu = \delta_0^\mu \mathcal{H}'^3_a dt. \quad (60)$$



The integrability condition ensures the independence between the parameters related to the involutive set of HJPDE. Therefore, since  $t = x^0$  is one of these parameters, we can analyse the temporal evolution of the fields independently. We have

$$\partial_0 A_\mu^a = \delta_\mu^i (D_i A_0^a - \Lambda f_{bc}^a B_i^b B_0^c), \quad (61)$$

$$\partial_0 B_\mu^a = \delta_\mu^i (D_i B_0^a - f_{bc}^a A_0^b B_i^c). \quad (62)$$

Note that the component  $\mu = 0$  of these equations states that  $A_0^a, B_0^a$  are time independent parameters. This reinforce the character of lagrange multipliers of these variables in the canonical Hamiltonian. On the other hand, the spatial components of (61) are equivalent to the equation (31). Analogously, the spatial components of (62) resemble equations (32).

For the second set of CE, we have

$$\partial_0 \pi_a^\mu = \epsilon^{0\gamma\rho} [\delta_0^\mu D_\gamma B_{a\rho} - \delta_\rho^\mu (D_\gamma B_{a0} - f_a^{bc} A_{b0} B_{c\gamma})], \quad (63)$$

$$\partial_0 \Pi_a^\mu = \delta_0^\mu \mathcal{H}_a'^3. \quad (64)$$

Note that, the temporal evolution of the component  $\pi_a^\mu$  is equal to the Hamiltonian  $\mathcal{H}_2'^a = 0$ , leaving  $\pi_a^0$  undetermined, just as its correspondent conjugated variable  $A_0^a$ . For the component  $\pi_a^i$ , we have that its temporal evolution equation is in agreement with the definition of canonical momenta. For  $\Pi_a^\mu$ , we have that its temporal evolution is equal to zero. This result is in agreement with the fact that  $\Pi_a^0$  is conjugated to a Lagrange multiplier and the  $\Pi_a^i$  is no longer a canonical variable.

## 5.1 Generators of canonical and gauge transformations

As it was shown in section 2, the CE also give us the generator of the canonical transformations. In our case, we need to consider the variations along the independent parameters  $\omega_a$ .

$$dA_\mu^a = \delta_\mu^0 \delta^{ab} d\omega_b^0 - \delta_\mu^i \delta^{ab} D_i d\omega_b^2 - \delta_\mu^i \Lambda f_{bc}^a B_i^c d\omega_b^3, \quad (65)$$

$$dB_\mu^a = \delta_\mu^0 d\omega^{a1} - \delta_\mu^i f_{bc}^a B_i^b d\omega_c^2 - \delta_\mu^i \delta^{ab} D_i d\omega_b^3. \quad (66)$$

These expressions can be rewritten in a much simple form if we define the function

$$G^{can} \equiv \int [\mathcal{H}_0'^a d\omega_a^0 + \mathcal{H}_1'^a d\omega_a^1 + \mathcal{H}_2'^a d\omega_a^2 + \mathcal{H}_3'^a d\omega_a^3] d^2x. \quad (67)$$

It enables us to write

$$dA_\mu^a = \{A_\mu^a, G^{can}\}^*, \quad (68)$$

$$dB_\mu^a = \{B_\mu^a, G^{can}\}^*. \quad (69)$$

As the variations of the phase space coordinates can be expressed in this way, we call  $G^{can}$  the *generator of canonical transformations*.

On the other hand, in order to relate generator of canonical transformations with the one of symmetries, we need to go further the IC. Let us consider the set of variations (65) and (66) now rewritten as

$$\delta A_\mu^a = \delta_\mu^0 \delta^{ab} \delta\omega_b^0 - \delta_\mu^i \delta^{ab} D_i \delta\omega_b^2 - \delta_\mu^i \Lambda f_{bc}^a B_i^c \delta\omega_b^3, \quad (70)$$

$$\delta B_\mu^a = \delta_\mu^0 d\omega^{a1} - \delta_\mu^i f_{bc}^a B_i^b \delta\omega_c^2 - \delta_\mu^i \delta^{ab} D_i \delta\omega_b^3, \quad (71)$$

where the variations  $\delta\omega_a^\kappa$  may depend on each other. If the variations (70),(71) are symmetries of the three-dimensional BF gravity, they must be solutions of the fixed point variation

$$\delta\mathcal{L} = \frac{1}{2}\epsilon^{\alpha\mu\nu} (F_{\mu\nu}^a - \Lambda f_{bc}^a B_\mu^b B_\nu^c) \delta B_\alpha^a + \epsilon^{\alpha\mu\nu} B_\mu^a D_\nu \delta A_{a\alpha} = 0. \quad (72)$$

By replacing the (70),(71) in (72) and using the Bianchi identity it follows that

$$\begin{aligned} \delta\mathcal{L} = & \epsilon^{ij} \left[ \frac{1}{2} F_{ij}^a (\delta\omega^{a1} + f_{abc} B_0^b \delta\omega_c^2) + B_i^a D_j (D_0 \delta\omega_a^2 + \delta\omega_a^0) + F_{a0j} D_i \delta\omega^{a3} \right] \\ & + -\Lambda f_{abc} \epsilon^{ij} \left[ \frac{1}{2} B_i^b B_j^c (\delta\omega^{a1} + f_{nm}^a B_0^n \delta\omega^{m2}) - B_0^a B_j^c D_i \delta\omega^{b3} \right] \\ & + -\Lambda f_{abc} \epsilon^{ij} [-B_0^a D_i (B_j^b \delta\omega^{c3}) + B_i^a D_0 (B_j^b \delta\omega^{c3})]. \end{aligned} \quad (73)$$

Since this is one equation for four parameters, we expect to obtain a relation between some of the  $\delta\omega_a^\kappa$ . A good approach to solve  $\delta\mathcal{L} = 0$  is by considering special cases, as setting some of the parameters equal to zero. However, by inspection of (73), we see that  $\delta\omega^{a0} = 0$  or  $\delta\omega^{a1} = 0$  are not good choices for solving the equation. On the other hand, if we consider  $\delta\omega^{a3} = 0$ , equation (73) becomes

$$\begin{aligned} \delta\mathcal{L} = & \epsilon^{ij} \left[ \frac{1}{2} F_{ij}^a (\delta\omega^{a1} + f_{abc} B_0^b \delta\omega_c^2) + B_i^a D_j (D_0 \delta\omega_a^2 + \delta\omega_a^0) \right] \\ & + -\Lambda f_{abc} \epsilon^{ij} \left[ \frac{1}{2} B_i^b B_j^c (\delta\omega^{a1} + f_{nm}^a B_0^n \delta\omega^{m2}) \right]. \end{aligned} \quad (74)$$

Of course, we have an invariance,  $\delta\mathcal{L} = 0$ , when we choose  $\delta\omega_a^0 = -D_0 \delta\omega_a^2$  and  $\delta\omega^{a1} = -f_{bc}^a B_0^b \delta\omega^{c2}$ . By replacing it in the set of HJ variations, we get

$$\begin{aligned} \delta B_\mu^a &= -f_{bc}^a B_\mu^b \delta\omega^{c2}, \\ \delta A_{a\mu} &= -D_\mu \delta\omega_a^2. \end{aligned}$$

By setting  $\omega^{a2} = -\chi^a$ , we obtain the gauge transformation (27). These transformations are generated by

$$G^{Gauge} \equiv \int [\mathcal{H}_0'^a D_0 + \mathcal{H}_1'^b f_{bc}^a B_0^c - \mathcal{H}_2'^a] \delta\chi_a^2 d^2x. \quad (75)$$

Now, if we set  $\delta\omega^{a2} = 0$ , we obtain

$$\begin{aligned} \delta\mathcal{L} = & \epsilon^{ij} \left[ \frac{1}{2} F_{ij}^a \delta\omega^{a1} + B_i^a D_j \delta\omega_a^0 + F_{a0j} D_i \delta\omega^{a3} \right] \\ & + -\Lambda f_{abc} \epsilon^{ij} \left[ \frac{1}{2} B_i^b B_j^c \delta\omega^{a1} - B_0^a D_i (B_j^b \delta\omega^{c3}) + B_i^a D_0 (B_j^b \delta\omega^{c3}) - B_0^a B_j^c D_i \delta\omega^{b3} \right], \end{aligned}$$

which, up to boundary terms, becomes

$$\begin{aligned} \delta\mathcal{L} = & \epsilon^{ij} \left[ -D_j B_i^a (\delta\omega_a^1 - \Lambda f_{abc} B_0^b \delta\omega^{c3}) + \frac{1}{2} F_{aij} (D_0 \delta\omega^{a3} + \delta\omega^{a0}) \right] \\ & + -\epsilon^{ij} \Lambda f_{abc} \left[ \frac{1}{2} B_i^a B_j^b (D_0 \delta\omega^{c3} + \delta\omega^{c0}) \right]. \end{aligned} \quad (76)$$

This variation is equal to zero if we set  $\delta\omega^{a0} = -D_0\delta\omega^{a3}$  and  $\delta\omega_a^1 = \Lambda f_{abc}B_0^b\delta\omega^{c3}$ . Under these condition

$$\begin{aligned}\delta B_\mu^a &= -D_\mu\delta\omega^{a3}, \\ \delta A_{a\mu} &= \Lambda f_{abc}B_\mu^b\delta\omega^{c3},\end{aligned}$$

The shift transformation (28) can be obtained just by setting  $\eta^a = -\omega^{a3}$  in the previous relations. Its correspondent generator is given by

$$G^{shift} \equiv \int [\mathcal{H}_0'^a D_0 - \Lambda f_{bc}^a \mathcal{H}_1'^b B_0^c - \mathcal{H}_3'^a] \delta\eta_a^3 d^2x. \quad (77)$$

Therefore, we have obtained the gauge and shift transformations as well as its respective generators with the use of the HJ formalism.

## 6 Final Remarks

We have used the Hamilton-Jacobi formalism to analyse the constraint structure of the three-dimensional BF gravity with a cosmological constant  $\Lambda$ . This procedure consisted in finding the complete set of involutive Hamiltonians that generates the dynamical evolution of the system. We achieved this using the Frobenius' Integrability Condition over the initial set of HJPDE. We noticed that there is a subgroup of Hamiltonians  $(\mathcal{A}_a'^1, \mathcal{A}_a'^2, \mathcal{B}_1'^a, \mathcal{B}_2'^a)$  that does not satisfy the IC and with them we built the GB and reduced the phase space such that the system was governed by a new symplectic structure. By satisfying the IC for the rest of Hamiltonians, we found new constraints  $(\mathcal{C}'_a, \mathcal{D}'_a)$ . In the case of cosmological constant equal zero, these Hamiltonians satisfy the  $ISO(1, 2)$  algebra and the Hamiltonians  $\mathcal{D}'_a$  commute. When the cosmological constant is other than zero, the Hamiltonians satisfy the  $AdS$  or  $dS$  algebra.

Then, we computed the characteristic equations, which depend on the time parameter  $x^0$  and the parameters  $\omega_a^\kappa$  related to the involutive Hamiltonians. Since all the Hamiltonians satisfy the IC, their correspondent parameters are linearly independent. It means that evolution along any parameter can be considered independently. As a result, we saw that time evolution of the CE are equivalent to the field equations of BF gravity and the evolution along the parameters  $\omega_a^\kappa$  is related to the canonical transformations. Therefore, the linear combination of the four corresponding Hamiltonians gave the generator of the canonical transformations.

It was possible to relate the generator of canonical transformations with the one related to the gauge and shift transformations. To achieve this, we considered the  $\omega_a^\kappa$  parameters as dependent on each other. Furthermore, if they are an invariance of the theory they must eliminate, up to boundary term, the fixed point Lagrangian variation. This way, we needed to solve an equation for four dependent variables.

## 7 Acknowledgements

The authors thank M. C. Bertin for reading the manuscript and suggestions. N. T. Maia was supported by CAPES. B. M. Pimentel was partially supported by CNPq and CAPES. C. E. Valcárcel was supported by FAPESP.

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